

Note on Inversion Formula to Determine Binary Elements by Astrometry

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Abstract

Simplified solutions to determine binary elements by astrometry were obtained in terms of elementary functions (Asada et al. 2004), and therefore require neither iterative nor numerical methods. In the framework of the simplified solution, this paper discusses the remaining two parameters of the time of periastron passage and the longitude of ascending node in order to complete the solution. We thus clarify a difference between the simplified solution and other analytical methods.

Key words: astrometry — celestial mechanics — stars: binaries: general

1. Introduction

Recently, we developed a formulation for determining binary elements with astrometric observations. The simplified solution is written in terms of elementary functions, and therefore requires neither iterative nor numerical methods (Asada et al. 2004). This solution has been generalized to a binary system in open (hyperbolic or parabolic) orbits as well as closed (elliptic) ones (Asada 2007). An extension to observational data has been also discussed (Asada et al. 2007). The solution gives an explicit form of binary elements such as the eccentric anomaly and the major axis of elliptic orbits. Oyama et al. (2008) made an attempt to use this solution for discussing some uncertainty in binary elements because of large scatter of their data points, when they measured proper motions of maser sources in the galactic center with VERA.

On the other hand, the remaining parameters of the time of periastron passage and the longitude of ascending node are not discussed in the simplified solution. Hence, the solution is rather simplified. However, these parameters are needed to make a comparison between the simplified solution and conventional ones. In addition, the lack of information on the remaining parameters apparently suggests a certain incompleteness of the simplified solution. In this brief article, therefore, we shall derive, in the framework of the simplified solution, both the time of periastron passage and the longitude of ascending node in order to complete the solution.

Astrometry plays a fundamental role in astronomy through providing useful star catalogs based on precise measurements of the positions and movements of stars and other celestial bodies. For instance, astrometric observations provide an useful method of determining mass of various unseen celestial objects currently such as a massive black hole (Miyoshi et al. 1995), an extra-solar planet (Benedict et al. 2002) and two new satellites of Pluto (Weaver et al. 2006). Astrometry of Sharpless 269 with VERA detects a trigonometric parallax corresponding to a distance of 5.28 kpc, which is the smallest parallax ever measured, and puts the strongest constraint on the flatness of outer rotation curve (Honma et al. 2007). Accordingly, astrometry has attracted renewed interests, since the Hipparcos mission successfully provided us the precise catalog at the level of a milliarcsec. In fact, there exist several projects of space-borne astrometry aiming at a accuracy of a few microarcseconds, such as SIM¹ (Shao 2004), GAIA² (Mignard 2004, Perryman 2004) and JASMINE³ (Gouda et al. 2007).

In this paper, we focus on an astrometric binary, for which only one of the component stars can be visually observed but the other cannot, like a black hole or a very dim star. In this case, it is impossible to directly measure the relative vector connecting the two objects, because the secondary is not directly observed. The position of the star is repeatedly measured relative to reference stars or quasars. On the other hand, the orbit determination of resolved double stars (visual binaries), which are a system of two visible stars, was solved first by Savary in 1827 and by many authors including Kowalsky, Thiele and Innes (Binnendijk 1960, Aitken 1964 for a review on earlier works; for the state-of-the-art techniques, e.g., Eichhorn and Xu 1990, Catovic and Olevic 1992, Olevic and Cvetkovic 2004). The relative vector from the primary star to the secondary has an elliptic motion with a focus at the primary. This relative vector is observable only for resolved double stars.

In conventional methods of orbit determination, the time of periastron passage is one of important parameters because it enters the Kepler's equation as

$$t = t_0 + \frac{T}{2\pi}(E - e_K \sin E), \quad (1)$$

where t_0 , T , e_K and E denote the time of periastron passage, orbital period, eccentricity and eccentric anomaly, respectively (e.g., Danby 1988, Roy 1988, Murray and Dermott 1999, Beutler 2004). The simplified solution does not use the Kepler's equation in order to avoid treating such a transcendental equation.

This paper is organized as follows. Our notation in the simplified solution will be summarized in § 2. The time of periastron passage in the simplified solution will be derived in § 3. The longitude of ascending node will be obtained in § 4.

¹ <http://sim.jpl.nasa.gov/>

² <http://www.rssd.esa.int/index.php?project=GAIA&page=index>

³ <http://www.jasmine-galaxy.org/>

2. Simplified solution: Our notation

Our notation in the simplified solution is briefly summarized as follows. We neglect motions of the observer and the common center in our galaxy. Namely, we take account only of the Keplerian motion of a star around the common center of mass of a binary system. Let us define (x, y) as the Cartesian coordinates on a celestial sphere, in such a way that the apparent (observed) ellipse on the celestial sphere can be expressed in the standard form as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad (2)$$

where $a \geq b$. The eccentricity e is $\sqrt{1 - b^2/a^2}$. This eccentricity may be different from e_K , the eccentricity of the actual elliptic orbit, because of the inclination of the orbital plane with respect to the line of our sight. The star is located at $P_j = (x_j, y_j)$ on the celestial sphere at the time of t_j for $j = 1, \dots, n$.

We use a fact that the law of constant-areal velocity still holds, even after a Keplerian orbit is projected onto the celestial sphere. Here, the area is swept by the line interval between the star and the projected common center of mass but not a focus of the apparent ellipse (See Fig. 1). This fact is expressed as

$$\frac{S}{T} = \frac{S(k, j)}{T(k, j)}, \quad (3)$$

where $S(k, j)$ and S denote the area swept during the time interval, $T(k, j) = t_k - t_j$ for $t_k > t_j$, and the total area of the apparent ellipse πab , respectively. The swept area is expressed as (Asada et al. 2004, Asada 2007)

$$S(k, j) = \frac{1}{2}ab \left[u_k - u_j - \frac{x_e}{a}(\sin u_k - \sin u_j) + \frac{y_e}{b}(\cos u_k - \cos u_j) \right]. \quad (4)$$

The eccentric anomaly in the apparent ellipse is given by $u_j = \arctan(ay_j/bx_j)$.

The orbital elements can be expressed explicitly as elementary functions of the locations of four observed points and their time intervals (Asada et al. 2004). Let us take four observed points P_1, P_2, P_3 and P_4 for $t_1 < t_2 < t_3 < t_4$. The location (x_e, y_e) of the projected common center is given by

$$x_e = -a \frac{F_1 G_2 - G_1 F_2}{E_1 F_2 - F_1 E_2}, \quad (5)$$

$$y_e = b \frac{G_1 E_2 - E_1 G_2}{E_1 F_2 - F_1 E_2}, \quad (6)$$

where E_j, F_j and G_j are elementary functions of $T(j+2, j+1), T(j+1, j)$ and u_k for $k = j, j+1, j+2$. The eccentric anomaly in the actual ellipse (on the orbital plane) is denoted as E (See Eq. (1)).

Given a, b, x_e and y_e , we can analytically determine the parameters e_K, i, a_K and ω as (Asada et al. 2004)

$$e_K = \sqrt{\frac{x_e^2}{a^2} + \frac{y_e^2}{b^2}}, \quad (7)$$

$$\cos i = \frac{1}{2}(\xi - \sqrt{\xi^2 - 4}), \quad (8)$$

$$a_K = \sqrt{\frac{C^2 + D^2}{1 + \cos^2 i}}, \quad (9)$$

$$\cos 2\omega = \frac{C^2 - D^2}{a_K^2 \sin^2 i}, \quad (10)$$

where

$$C = \frac{1}{e_K} \sqrt{x_e^2 + y_e^2}, \quad (11)$$

$$D = \frac{1}{abe_K} \sqrt{\frac{a^4 y_e^2 + b^4 x_e^2}{1 - e_K^2}}, \quad (12)$$

$$\xi = \frac{(C^2 + D^2) \sqrt{1 - e_K^2}}{ab}. \quad (13)$$

3. Time of periastron passage

In order to determine a_K and e_k for an actual ellipse, the simplified solution requires neither the time of periastron passage t_0 nor the longitude of ascending node, Ω (Asada et al. 2004). If one wishes to know t_0 and Ω , however, they can be determined as follows (See also Fig. 2). First, we discuss t_0 in this section.

The projected position of the periastron on the celestial sphere, \mathbf{P}_A , is determined as

$$\mathbf{P}_A = \frac{1}{e_K} (x_e, y_e), \quad (14)$$

because the ratio of the semimajor axis to the distance between the center and the focus of the ellipse remains unchanged, even after the projection (Asada et al. 2004). The eccentric anomaly u_A of the periastron in the apparent ellipse is introduced as

$$\mathbf{P}_A = (a \cos u_A, b \sin u_A), \quad (15)$$

where \mathbf{P}_A is given also by Eq. (14). Thereby, we can determine $u_A \pmod{2\pi}$.

By using Eq. (3), we obtain

$$\frac{S(1,0)}{T(1,0)} = \frac{S(2,1)}{T(2,1)}, \quad (16)$$

where we can determine $S(1,0)$ because the eccentric anomaly in the apparent ellipse at t_0 , denoted as u_0 , is nothing but u_A , which has been determined by Eqs. (14) and (15). Therefore, Eq. (16) is solved for t_0 as

$$t_0 = \frac{S(2,0)}{S(2,1)} t_1 - \frac{S(1,0)}{S(2,1)} t_2. \quad (17)$$

where the R.H.S. is obtained from observed quantities.

4. Longitude of ascending node

Let us consider the projected periastron at \mathbf{P}_A on the apparent ellipse. In the simplified solution, \mathbf{P}_A is expressed as Eq. (14). Here we make a translation of (x, y) in such a way that the common center of mass can be located at the origin of new coordinates (x', y') . Namely, the x' axis is taken to lie along the major axis of the apparent ellipse in the celestial sphere, and the y' axis is perpendicular to the x' axis in the celestial sphere (See Fig. 3). In the coordinates (x', y') , the position of the projected periastron becomes

$$\mathbf{P}_A = \frac{1 - e_K}{e_K} (x_e, y_e). \quad (18)$$

On the other hand, by projecting the actual ellipse onto the celestial sphere, we obtain

$$\mathbf{P}_A = (a_K(1 - e_K) \cos \omega, a_K(1 - e_K) \sin \omega \cos i), \quad (19)$$

where the coordinates (\bar{x}, \bar{y}) are chosen so that the ascending node can be in the \bar{x} -direction (See Fig. 3).

The longitude of ascending node, which is the angle between the \bar{x} and x' axes, relates the two coordinates of (x', y') and (\bar{x}, \bar{y}) by rotation. Therefore, from Eqs. (18) and (19), we obtain

$$\begin{pmatrix} \cos \Omega & -\sin \Omega \\ \sin \Omega & \cos \Omega \end{pmatrix} \begin{pmatrix} a_K(1 - e_K) \cos \omega \\ a_K(1 - e_K) \sin \omega \cos i \end{pmatrix} = \frac{1 - e_K}{e_K} \begin{pmatrix} x_e \\ y_e \end{pmatrix}. \quad (20)$$

This relation determines Ω ($\text{mod } 2\pi$). For instance, we obtain explicitly

$$\tan \Omega = \frac{y_e \cos \omega - x_e \sin \omega \cos i}{y_e \sin \omega \cos i + x_e \cos \omega}, \quad (21)$$

where x_e , y_e , i and ω in the R.H.S. have been determined by Eqs. (5), (6), (8) and (10).

In conventional methods, determining Ω is tightly coupled with ω and i . On the other hand, it can be done separately from ω and i in the simplified solution.

It should be noted that in practical applications a reference direction chosen by observers may be different from the major axis of the apparent ellipse. In such a practical case, Ω is the angle from the reference direction to the direction of the ascending node. To compute the longitude of ascending node, therefore, the angle $\delta\Omega$ from the reference direction to the major axis is added into the angle measured from the major axis. In short, the longitude of ascending node is generally the sum of Ω_0 and $\delta\Omega$, where Ω_0 is the angle Ω determined by using Eq. (21). The expression of $\delta\Omega$ is obtained in the straightforward manner, for instance as Eq. (6) in Asada et al. (2007), where they denoted $\delta\Omega$ as Ω .

Tables 1 and 2 give an example to show the flow of actual determination of all six orbital elements. This would be helpful to the readers who code computing routines in practical applications to check their programs. Table 1 shows some given values for all six orbital elements and the orbital period. Based on these elements, we first prepare a virtual observation data

set, which is listed in Table 2. The elements are then reproduced from the data by using the proposed method.

Here, we discuss how to determine in the simplified solution a position of a component star at arbitrary time $t \equiv t_n \pmod{T}$. For t_1 , t_2 and t_n , Eq. (3) becomes

$$\frac{S(n,1)}{T(n,1)} = \frac{S(2,1)}{T(2,1)}. \quad (22)$$

This is a transcendental equation for u_n on the celestial sphere. This situation seems similar to that the Kepler's equation is transcendental in E on the orbital plane. Here we should note that the time of periastron passage is needed in order to treat Kepler's equation, whereas it is not for Eq. (22). This is because we employ the time interval $T(k,j)$, while the Kepler's equation needs the time itself instead of the interval. Regarding this point, Thiele's method for visual binaries is closer to the simplified solution, in the sense that they use the time interval in order to delete the time of periastron passage. A crucial difference is that Thiele's method uses Kepler's equation on the orbital plane (Thiele 1883), while the simplified one does the constant areal velocity in the apparent ellipse on the celestial sphere. In this sense, the simplified solution more respects measured quantities on the celestial sphere than conventional ones (See Fig. 2). It is verified numerically that the above procedure enables us to determine in the simplified solution locations of a star at arbitrary time (See Fig. 4 for an example).

5. Conclusion

In this paper, we obtain in the simplified solution both the time of periastron passage and the longitude of ascending node in order to complete the solution (See Fig. 2). In conclusion, the simplified solution requires neither iterative nor numerical methods when we determine all the elements including t_0 and Ω . It does only when we wish to determine the star's position at arbitrary time.

Before closing this paper, it is worthwhile to mention that Eqs. (17), (21) and (22) can be applied to a case of open orbits in the straightforward manner. For open orbits, expressions of x_e , y_e , e_K , a_K , i and ω have been already derived in the framework of the simplified solution (Asada 2007).

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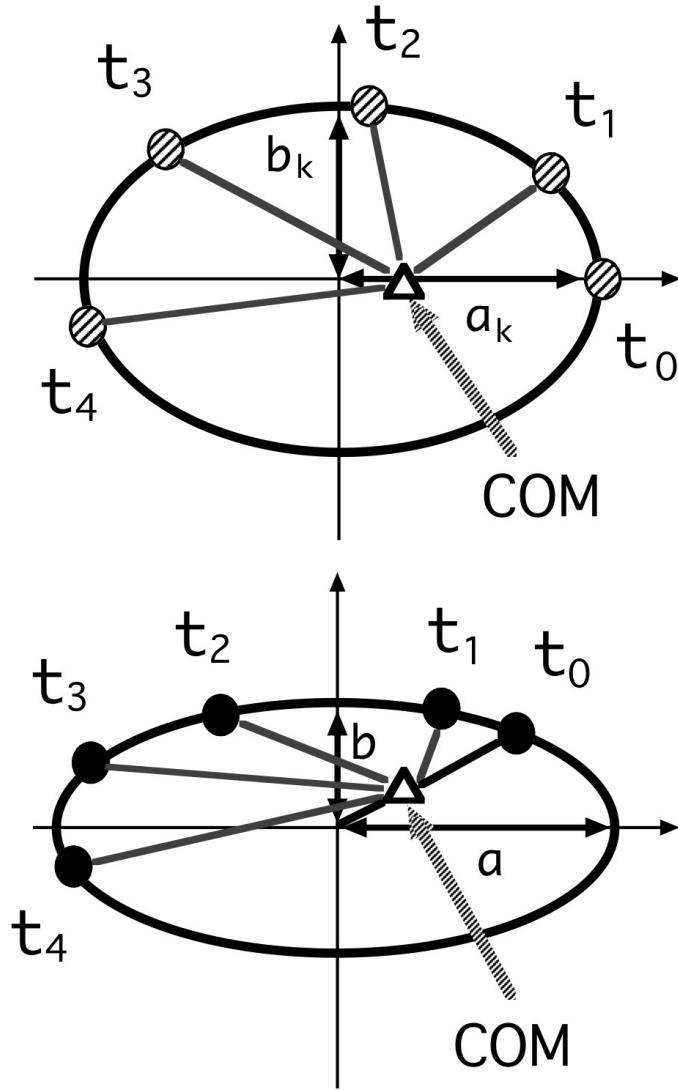


Fig. 1. Schematic figures of the actual elliptical orbit (top figure) and the apparent one (bottom one). The semimajor and semiminor axes of the actual elliptical orbit are denoted by a_k and b_k , respectively. Those of the apparent one are a and b , respectively. The position of the star at each time is denoted by shaded circles (in the top figure) and filled circles (in the bottom figure). The triangles indicate the center of mass (COM). In the bottom figure, the projected center of mass is located on the line connecting the center of the apparent ellipse and the projected periastron (at t_0).

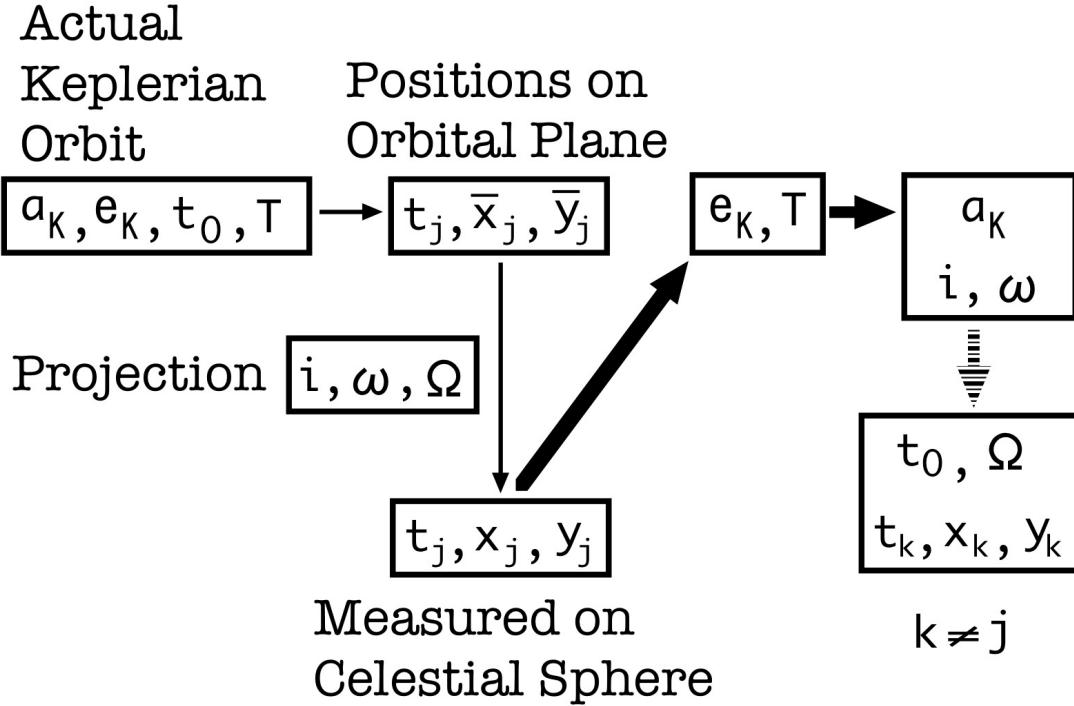


Fig. 2. Flow chart of our procedure of orbit determination. The thin arrow denotes purely theoretical steps, where we initially assume an actual Keplerian orbit parameterized by (a_K, e_K, t_0, T) . A star's position on the orbital plane at each time t_j is projected onto the celestial sphere defined by i, ω and Ω . The thick arrow denotes observational steps, where we start from measuring star's positions on the celestial sphere as (t_j, \mathbf{x}_j) . The steps of determining e_K, a_K, T and (i, ω) have been examined (Asada et al. 2004, Asada 2007). The remaining steps of computing t_0, Ω and (t_k, \mathbf{x}_k) in the simplified solution, denoted by the dashed arrow, are discussed in this paper.

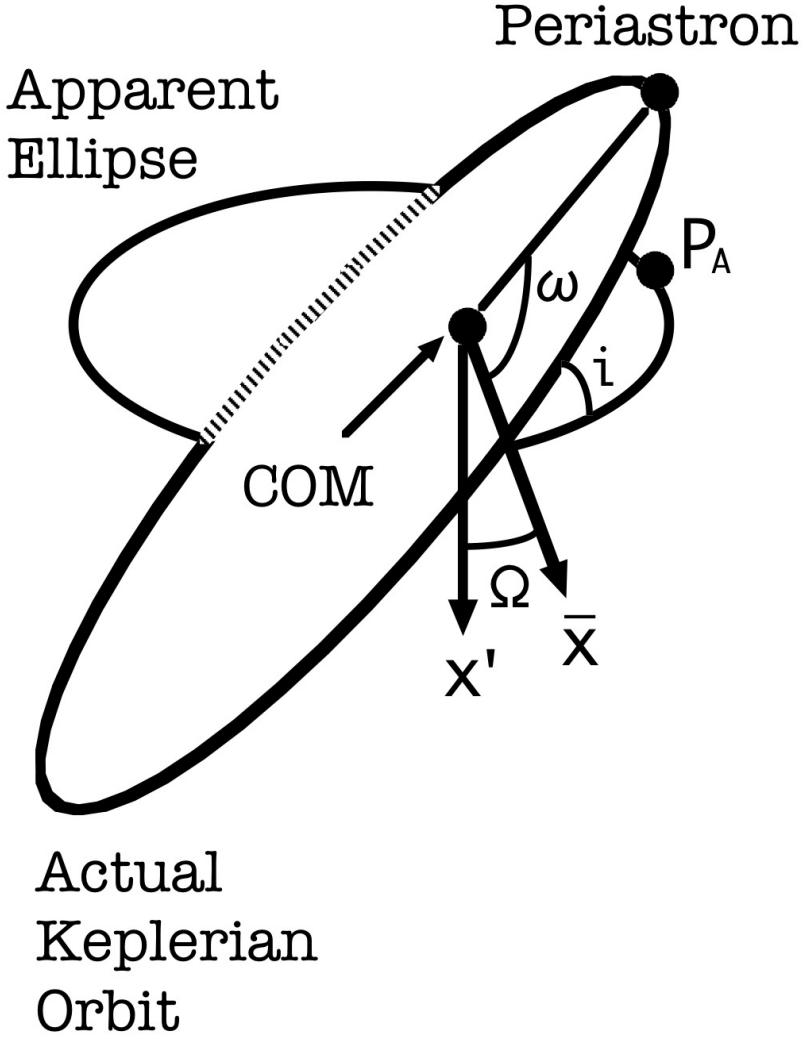


Fig. 3. Actual Keplerian orbit and apparent ellipse in three-dimensional space. We introduce the inclination angle i , the argument of periastron ω and the longitude of ascending node Ω . These angles relate two coordinates (x', y') and (\bar{x}, \bar{y}) , both of which choose the origin as the common center of mass. Here the x' axis is taken to lie along the major axis of the apparent ellipse, while the \bar{x} -axis is taken to lie along the direction of the ascending node.

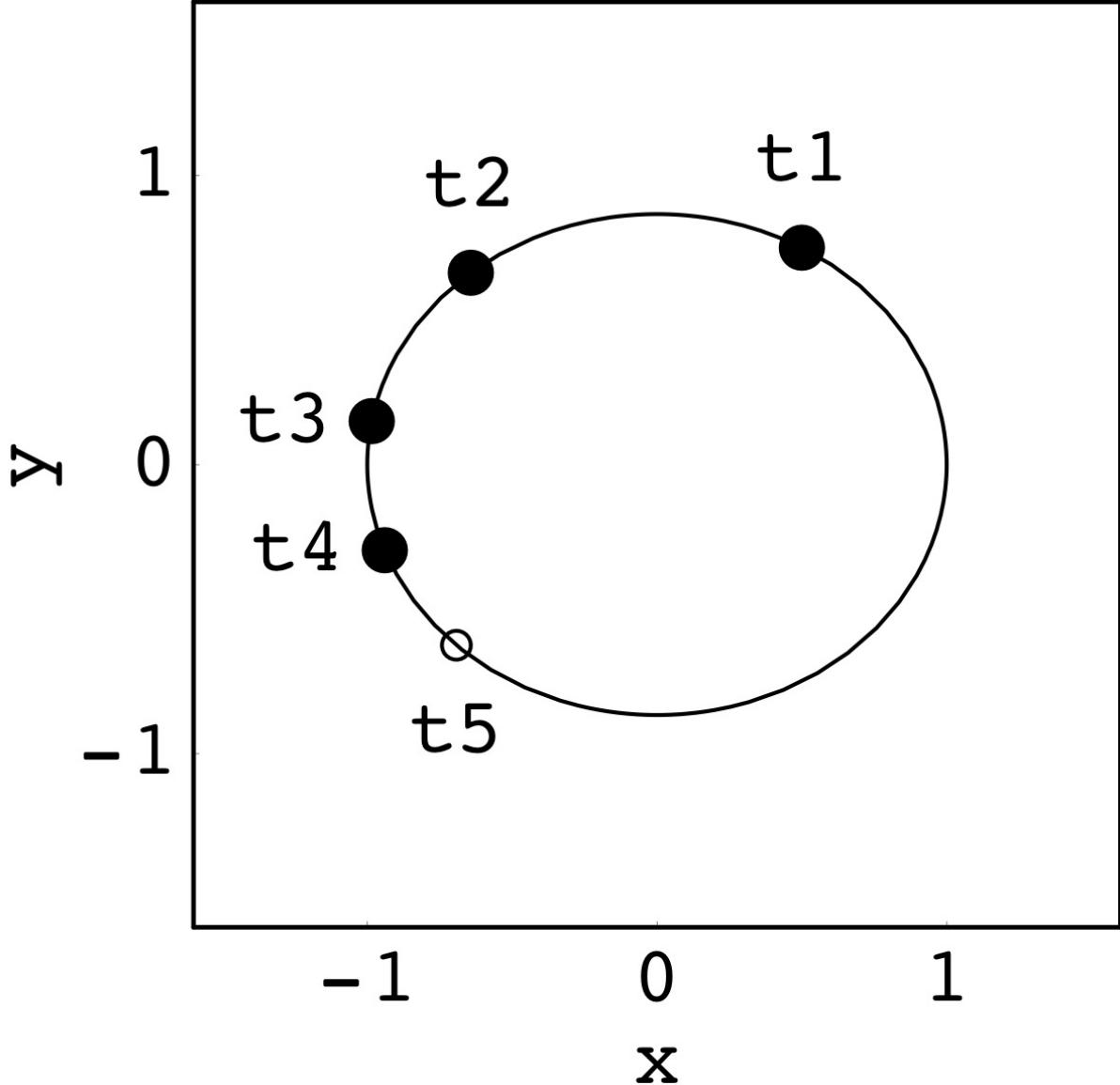


Fig. 4. Example of orbit determination. Here we assume $a = 1$, $e = 0.5$ as an apparent ellipse on the celestial sphere. Let a star located at $u_1 = 60$, $u_2 = 130$, $u_3 = 170$, $u_4 = 200$ (deg.) at each time t_i for $i = 1, 2, 3, 4$. Regarding time, we assume that the time interval $T(i+1, i)$ is the same as unity, namely $t_i = i - 1$ (i.e. $t_1 = 0$), for simplicity. The simplified solution allows for arbitrary time interval. The observed positions of the star are denoted by the filled circle. The quantities determined in the present procedure are $e_K = 0.56$, $a_K = 1.2$, $i = 42$ deg. and $\omega = 79$ deg. We obtain $t_0 = 0.04$ by Eq. (17). The star's position at $t_5 = 5$ is obtained as $u_5 = 2.3 \times 10^2$ deg. The location is denoted by the circle. Using the determined e_K , a_K , i and ω , we determine u_5 at t_5 also by employing the conventional procedure (indicated by the thin arrow in Fig. 2). In the latter case, we need to take account of the longitude of ascending node, Ω . We obtain $\Omega = 15$ (deg.) by Eq. (21). The results of u_5 by both methods agree with each other.

Table 1. Numerical example of orbital elements. The reference direction is taken along the x -axis, which is different from the major axis of the apparent ellipse in this example. Ω is the angle measured from the reference direction.

a_k	e_k	i	ω	Ω	t_0	T
1	0.5	$\pi/8$	$\pi/9$	$\pi/10$	0	20

Table 2. Virtual observational data set based on the orbital elements in Table 1. For simplicity, we assume that observations are sampled with the same frequency.

$t_j = j$	x_j	y_j
1	0.372003	0.838658
2	-0.0648698	0.831542
3	-0.404177	0.696231
4	-0.646827	0.509083
5	-0.809209	0.304280